Maximum Time: 3 hours
Maximum Score: 100

If you use a result proven in class then please state it clearly and verify the hypothesis while using the same.

1. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables.
   (a) **(10 points)** Suppose that

   $$E[X_n] = 0 \qquad \text{and} \qquad 0 \leq E[X_n^2] \leq 1 \text{ for all } n \geq 1.$$

   Show that for any $\alpha > \frac{1}{2}$, $\frac{1}{n^\alpha}\sum_{i=1}^n X_i$ converges to 0 in probability.

   (b) Suppose $X_n \sim X$
      i. **(5 points)** Show that $\frac{X_n}{n}$ converges to 0 in probability.
      ii. **(10 points)** Provide necessary and sufficient conditions for $\frac{X_n}{n}$ to converge to 0 with probability 1.

2. Let X_n be a Markov chain on S with transition matrix P and initial distribution μ. .
   (a) **(5 points)** Let A be an event. Does it necessarily imply that

   $$\mathbb{P}(X_n = j \mid X_{n-1} \in A, X_{n-2} = i_{n-2}, \ldots, X_0 = i_0) = \mathbb{P}(X_n = j \mid X_{n-1} \in A)$$

   (b) Suppose $S = \mathbb{Z}$, the tranistion matrix P given by

   $$p_{ij} = \begin{cases} p & \text{if } j = i+1 \\ 1-p & \text{if } j = i-1 \\ 0 & \text{otherwise,} \end{cases}$$

   where $0 < p < 1$, and suppose $\mu(\{0\}) = 1$.
      i. **(10 points)** Find the best possible $\mu(a,p)$ such that

      $$\mathbb{P}(X_n \geq na) \leq \exp(-n\mu(a,p)) \text{ for all } a > 2p-1.$$

   (c) Let S be $\{0, 1, 2, \ldots, L\}$, the transition matrix $P = [p_{ij}]$ be given by

   $$p_{ij} = \begin{cases} 1 & \text{if } j = 0, i = 0, \text{ and } j = L, i = L. \\ p & \text{if } j = i+1, i \neq 0, i \neq L, \\ 1-p & \text{if } j = i-1, i \neq 0, i \neq L, \\ 0 & \text{otherwise.} \end{cases}$$

      i. **(5 points)** Decide if the chain is irreducible.
      ii. **(5 points)** For each $i \in S$, find the period of i.
      iii. **(10 points)** Compute $h : S \to [0,1]$ given by $h(i) = \mathbb{P}(\sigma_{\{0\}} < \sigma_{\{L\}} \mid X_0 = i)$. with $\sigma_m = \inf\{k \geq 0 : X_k = m\}$.

3. Let $X, \{X_n\}_{n\geq 1}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
   (a) **(5 points)**Suppose X_n is a Gamma $(\frac{n}{2}, \frac{1}{2})$, then show that $\frac{X_n - n}{\sqrt{n}}$ converges in distribution to a standard Normal random variable.

(b) **(10 points)** Suppose X has a probability density funciton $f : \mathbb{R} \to [0, \infty)$. Show that X_n converges in distribution to X if and only if $\mathbb{P}(a < X_n \leq b) \to \mathbb{P}(a < X \leq b)$ as $n \to \infty$ for all $a, b \in \mathbb{R}$.

(c) **(10 points)** Suppose X_n converges to X in distribution as $n \to \infty$ then show that X_n is tight.

4. Let $N > 0$ be a fixed natural number. Let

$$\Omega_N = \{\omega = (\omega_1, \omega_2, \ldots, \omega_N) : \omega_i \in \{-1, 1\}\} \equiv \{-1, 1\}^N$$

and $\mathcal{A}_N$ is the collection of all subsets of Ω_N. Define $P : \mathcal{A}_N \to [0, 1]$, by

$$P(A) = \frac{\mid A \mid}{2^N}.$$

For $1 \leq k \leq N$, let $X_k : \Omega_N \to \{-1, 1\}$ given by $X_k(\omega) = \omega_k$ denote the displacement in the k-th step of the walk and for $1 \leq n \leq N$ let

$$S_n(\omega) = \sum_{k=1}^{n} X_k(\omega),$$

denote the position of the random walk at time n.

(a) **(5 points)** Show that P is a probability on $(\Omega_N, \mathcal{A}_N)$.

(b) **(5 points)** Show that $P(X_k = 1) = P(X_k = -1) = \frac{1}{2}$ for all $1 \leq k \leq n$ and that $X_1, X_2, \ldots X_N$ are independent.

(c) **(5 points)** Suppose $0 < k_1 < k_2 < k_3 < N$. Show that $S_{k_2} - S_{k_1}$ and $S_{k_3} - S_{k_2}$ are independent.

(d) **(5 points)** Suppose for $0 < k < m < N$, $a, b \in \mathbb{Z}$ we have $P(S_k = a) > 0$ then show that

$$P(S_m = b \mid S_k = a) = P(S_{m-k} = b - a).$$